

Harmonic Oscillator on the $SO(2, 2)$ Hyperboloid[★]

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Abstract. In the present work the classical problem of harmonic oscillator in the hyperbolic space H_2^2 : $z_0^2 + z_1^2 - z_2^2 - z_3^2 = R^2$ has been completely solved in framework of Hamilton–Jacobi equation. We have shown that the harmonic oscillator on H_2^2 , as in the other spaces with constant curvature, is exactly solvable and belongs to the class of maximally superintegrable system. We have proved that all the bounded classical trajectories are closed and periodic. The orbits of motion are ellipses or circles for bounded motion and ultraellipses or equidistant curve for infinite ones.

Key words: superintegrable systems; harmonic oscillator; hyperbolic space; Hamilton–Jacobi equation

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1 Introduction

The harmonic oscillator as a distinguished dynamical system plays the fundamental role in theoretical and mathematical physics due to many special properties outgoing from its hidden symmetry. Together with the Kepler–Coulomb problem they are only one among the central potentials for which all classical trajectories are closed (Bertrand theorem) and in quantum mechanics all energy state are multiply degenerate (accidental degeneracy). The other consequence of hidden symmetry is the existence of additional functionally (in quantum mechanics linearly) independent integrals of motion and the phenomena of multiseparability, that is separability of variables in Hamilton–Jacobi or Schrödinger equation in more than one orthogonal systems of coordinate. It has long been known [9, 12, 26] that the harmonic oscillator problem possesses five functionally independent integrals of motion, which generate the separation of variable into eight systems of coordinates [11, 17]. In most of them harmonic oscillator admits the exact solution, the fact which makes it attractive to use as a model of molecular, atomic and nuclear physics and other branches of theoretical physics.

The generalization of Kepler–Coulomb system and oscillator problem on the spaces of constant curvature start from the work of Lobachevsky, who first identified the Kepler potential in hyperbolic space H_3 (two-sheeted hyperboloid) and found the trajectories of classical motion [38] (see also the articles [7, 10, 34, 47]). The extension of the harmonic oscillator problem on the

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spherical and hyperbolic geometries has already been done in the book of Liebmann [37], who also discussed the geometric character of the conics in noneuclidean geometry. The investigation of Kepler–Coulomb problem in quantum mechanics was motivated to compare the properties of the Coulomb potential in the “open hyperbolic” or “closed” universe to that of an “open but flat” universe. Schrödinger [46] was the first who discussed this problem and discovered that for “hydrogen atom” on three-dimensional sphere only discrete spectrum exists. Virtually at the same time, Infeld and Schild [25] found that in an open hyperbolic universe there is only a finite (but very large) number of bound states. The motion in Coulomb field on imaginary Lobachevsky space (one-sheet hyperboloid), as shown by Grosche [16], has some peculiarities. It is not singular for any value of variable and their discrete spectra infinite degenerate. The essential advance in the theory of systems with hidden symmetry in the spaces with constant curvature was made by Higgs [24], Leemon [36] and Belorussian authors in [2]. They have shown that the complete degeneracy of the spectrum of the Coulomb and oscillator problems on the three-dimensional sphere and hyperboloid is caused by an additional integrals of motion: “curved” Runge–Lenz’s vector (for the Coulomb potential) and Demkov–Fradkin tensor (for the oscillator). However, in contrast to the flat space, commutation relations between the components of Runge–Lenz’s vector and Demkov–Fradkin tensor on the sphere and hyperboloid form the quadratic or cubic algebra. Later it was proven that these properties are inherent in all class of maximally second-order superintegrable systems, which also belong to the Kepler–Coulomb and oscillator potentials (see for instance recent review [40] and references therein).

We recall that in general, in an N -dimensional space, maximal superintegrability means that the classical Hamiltonian allows $(2N - 1)$ functionally independent integrals of motion (including the Hamiltonian) that are well defined functions on phase space. The first search of superintegrable systems in two- and three-dimensional flat space was done in the pioneering works of Winternitz and Smorodinsky with co-authors in [39, 50], later the notion of superintegrability in the spaces of constant curvature has been introduced in the series of papers [17, 18, 19, 20]. The complete classification of superintegrable systems on the two-dimensional complex sphere, which include to real spaces, sphere and hyperboloid, as particular cases have been done in the work [28]. Some of the superintegrable systems have been constructed on S_N and H_N spaces in [23]. We can also mention some articles devoted to the investigation of various aspects of both classical and quantum superintegrable systems in the spaces of constant curvature, for instance [2, 21, 27, 30, 31, 32].

The classical and quantum mechanical systems on the spaces of constant curvature (positive and negative) have always drawn a great attention due to their connection with the relativistic physics and gravity. The 2D and 3D one-sheeted and $SO(2, 2)$ hyperboloids are the models of the relativistic spacetime with a constant curvature, namely de Sitter and anti de Sitter spaces, which is a crucial point for its wide application in the field theories [42, 49], quantum gravity and cosmology [1, 14, 48], integrable Yang–Mills–Higgs equation (or Bogomolny equation) [33, 51]. Among other applications we can mention also quantum Hall effect [3] and coherent state quantization [13].

However, as far as we know, the superintegrable systems on imaginary Lobachevski space H_2^1 : $SO(3, 1)/SO(2, 1)$, (de Sitter space time dS^{2+1}) on hyperboloid $H_2^2 = SO(2, 2)/SO(2, 1)$, (Anti de Sitter space time AdS^{2+1}), have not been studied with the same degree of detail and need to be further investigated. It appears that the first work in this direction (if we do not take into account the paper [16]) was the article [8] (see also more general case in [4]) where the authors, using the reduction procedure to the free Hamiltonian on the homogeneous space $SU(2, 2)/U(2, 1)$, obtain the eleven different types of maximally superintegrable systems on the hyperboloid H_2^2 . Later, in paper [22], the superintegrable generalization of harmonic oscillator and Kepler–Coulomb potentials covering the six three-dimensional spaces of constant curvature (including de Sitter and anti de Sitter spaces) in unified way, parametrized by two

contraction parameters defining the metric in each space, have been constructed. In these papers the classical superintegrable systems are only identified but have not been solved. Recently, also the main properties of two-dimensional harmonic oscillator problem have been investigated in [6], using again two parameters approach, in nine standard two-dimensional Cayley–Klein spaces, including the de Sitter dS^{1+1} and anti de Sitter AdS^{1+1} spaces.

The present work in a sense can be considered as a continuation of our previous articles [43, 44, 45], devoted to the investigation of classical and quantum Kepler–Coulomb problem and quantum harmonic oscillator problem on the configuration hyperbolic space with constant curvature H_2^2 . The given paper aims to investigate the harmonic oscillator problem on the whole hyperbolic space H_2^2 from the point of view of classical mechanics, which, to our knowledge, has not been elucidated in literature so far. This task seems more complicated but also more interesting than the analogous problem in the other three-dimensional hyperbolic spaces. It mainly derives from the complexity of the space H_2^2 which includes such subspaces as the one- and two-sheeted hyperboloids. This study will hopefully also help us to better understand the quantum case.

2 The hyperbolic space H_2^2 and constants of motion

A three-dimensional hyperboloid $H_2^2 \subset \mathbf{R}_{2,2}$ is described by the equation

$$z_0^2 + z_1^2 - z_2^2 - z_3^2 = R^2 \quad (2.1)$$

To be more specific we parametrize the hyperboloid (2.1) using the geodesic pseudo-spherical coordinate (r, τ, φ) [29, 43], namely

$$\begin{aligned} z_0 &= \pm R \cosh r, & z_1 &= R \sinh r \sinh \tau, \\ z_2 &= R \sinh r \cosh \tau \cos \varphi, & z_3 &= R \sinh r \cosh \tau \sin \varphi, \end{aligned} \quad (2.2)$$

where $r \geq 0$ is the “geodesic radial angle”, $\tau \in (-\infty, \infty)$, and $\varphi \in [0, 2\pi)$. The connection between two sets of coordinates $z_0 \rightarrow -z_0$ corresponds to the complex transformation of radial angle $r \rightarrow i\pi - r$. The system of coordinate (2.2) is valid only for $|z_0| \geq R$ and the missing part of the surface for $|z_0| < R$ may also be taken into account if we use another form of the pseudo-spherical coordinate

$$\begin{aligned} z_0 &= \pm R \cos \chi, & z_1 &= R \sin \chi \cosh \mu, \\ z_2 &= R \sin \chi \sinh \mu \cos \varphi, & z_3 &= R \sin \chi \sinh \mu \sin \varphi, \end{aligned} \quad (2.3)$$

where now $\chi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\mu \in (-\infty, \infty)$ and $\varphi \in [0, 2\pi)$. It is also easy to see that the two pseudo-spherical systems of coordinate (2.2) and (2.3) are connected by

$$r \rightarrow i\chi, \quad \tau \rightarrow \mu - i\pi/2. \quad (2.4)$$

Here we shall make use of the pseudo-spherical system of coordinate in form (2.2). To investigate the motion in the region $|z_0| \leq R$, everywhere below, we will use the transformation (2.4).

The restriction of the pseudo-euclidean metric $ds^2 = G_{\mu\nu} dz^\mu dz^\nu$, $G_{\mu\nu} = \text{diag}(-1, -1, 1, 1)$, $(\mu, \nu = 0, 1, 2, 3)$ on $\mathbf{R}_{2,2}$ to H_2^2 leads to the following formula

$$\frac{ds^2}{R^2} = dr^2 - \sinh^2 r d\tau^2 + \sinh^2 r \cosh^2 \tau d\varphi^2.$$

Then the kinetic energy is given by

$$\mathcal{T} = \frac{R^2}{2} (\dot{r}^2 - \sinh^2 r (\dot{\tau}^2 - \cosh^2 \tau \dot{\varphi}^2))$$

and the canonical momenta can be obtained in a usual way

$$p_r = \frac{\partial \mathcal{T}}{\partial \dot{r}} = R^2 \dot{r}, \quad p_\tau = \frac{\partial \mathcal{T}}{\partial \dot{\tau}} = -R^2 \sinh^2 r \dot{\tau}, \quad p_\varphi = \frac{\partial \mathcal{T}}{\partial \dot{\varphi}} = R^2 \sinh^2 r \cosh^2 \tau \dot{\varphi}.$$

Thus the free Hamiltonian in the pseudo-spherical phase space $(r, \tau, \varphi; p_r, p_\tau, p_\varphi)$ with respect to the canonical Lie–Poisson brackets

$$\{f, g\} = \sum_{i=1}^3 \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right), \quad (2.5)$$

has the form

$$\mathcal{H}_{\text{free}} = \frac{1}{2R^2} \left\{ p_r^2 - \frac{1}{\sinh^2 r} \left(p_\tau^2 - \frac{p_\varphi^2}{\cosh^2 \tau} \right) \right\}. \quad (2.6)$$

It is clear that isometry group of H_2^2 hyperboloid is given by $\text{SO}(2, 2)$ group. The corresponding Lie algebra is six dimensional. The generators of $\mathfrak{so}(2, 2)$ algebra can be written in terms of the ambient space $\mathbf{R}_{2,2}$ coordinates z_μ and momenta p_μ as

$$\begin{aligned} \mathcal{L}_1 &= -(z_2 p_3 - z_3 p_2), & \mathcal{L}_2 &= -(z_1 p_3 + z_3 p_1), & \mathcal{L}_3 &= (z_1 p_2 + z_2 p_1), \\ \mathcal{N}_1 &= (z_0 p_1 - z_1 p_0), & \mathcal{N}_2 &= -(z_0 p_2 + z_2 p_0), & \mathcal{N}_3 &= -(z_0 p_3 + z_3 p_0), \end{aligned} \quad (2.7)$$

and the Lie–Poisson brackets (2.5) with the help of three-dimensional metric $\bar{g}_{ik} = \text{diag}\{1, -1, -1\}$ reads

$$\{\mathcal{L}_i, \mathcal{L}_j\} = \bar{g}_{im} \bar{g}_{jn} \varepsilon_{mnk} \mathcal{L}_k, \quad \{\mathcal{N}_i, \mathcal{N}_j\} = \bar{g}_{im} \bar{g}_{jn} \varepsilon_{mnk} \mathcal{L}_k, \quad \{\mathcal{N}_i, \mathcal{L}_j\} = \bar{g}_{im} \bar{g}_{jn} \varepsilon_{mnk} \mathcal{N}_k,$$

where $i, j, k = 1, 2, 3$. There are two Casimir invariants, the first of which vanishes in realization (2.7):

$$\mathcal{C}_1 = \mathbf{L} \cdot \mathbf{N} = \mathbf{N} \cdot \mathbf{L} = \bar{g}_{ik} \mathcal{N}_i \mathcal{L}_k = \mathcal{N}_1 \mathcal{L}_1 - \mathcal{N}_2 \mathcal{L}_2 - \mathcal{N}_3 \mathcal{L}_3 = 0, \quad (2.8)$$

and the second one is

$$\mathcal{C}_2 = N^2 + L^2, \quad (2.9)$$

where

$$\begin{aligned} N^2 &= \mathbf{N} \cdot \mathbf{N} = \bar{g}_{ik} \mathcal{N}_i \mathcal{N}_k = \mathcal{N}_1^2 - \mathcal{N}_2^2 - \mathcal{N}_3^2, \\ L^2 &= \mathbf{L} \cdot \mathbf{L} = \bar{g}_{ik} \mathcal{L}_i \mathcal{L}_k = \mathcal{L}_1^2 - \mathcal{L}_2^2 - \mathcal{L}_3^2. \end{aligned} \quad (2.10)$$

The next step is computing the relationship between the ambient momenta and the geodesic polar one. Taking into account that four-dimensional canonical momentum p_μ ($\mu = 0, 1, 2, 3$)

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{z}^\mu} = G_{\mu\nu} \dot{z}^\nu, \quad \mathcal{L} = \frac{1}{2} G_{\mu\nu} \dot{z}^\mu \dot{z}^\nu,$$

where \mathcal{L} is a kinetic energy in the ambient space $\mathbf{R}_{2,2}$, we obtain that

$$\begin{aligned} R \cdot p_0 &= -R \cdot \frac{\partial z_0}{\partial t} = -\sinh r p_r, \\ R \cdot p_1 &= -R \cdot \frac{\partial z_1}{\partial t} = -\cosh r \sinh \tau p_r + \frac{\cosh \tau}{\sinh r} p_\tau, \end{aligned}$$

$$\begin{aligned}
R \cdot p_2 &= R \cdot \frac{\partial z_2}{\partial t} = \cosh r \cosh \tau \cos \varphi p_r - \frac{\sinh \tau \cos \varphi}{\sinh r} p_\tau - \frac{\sin \varphi}{\sinh r \cosh \tau} p_\varphi, \\
R \cdot p_3 &= R \cdot \frac{\partial z_3}{\partial t} = \cosh r \cosh \tau \sin \varphi p_r - \frac{\sinh \tau \sin \varphi}{\sinh r} p_\tau + \frac{\cos \varphi}{\sinh r \cosh \tau} p_\varphi.
\end{aligned}$$

Then the generators (2.7) in geodesic pseudo-spherical coordinates and momenta are given by the formulas

$$\begin{aligned}
\mathcal{N}_1 &= -\sinh \tau p_r + \cosh \tau \coth r p_\tau, \\
\mathcal{N}_2 &= -\cosh \tau \cos \varphi p_r + \coth r \sinh \tau \cos \varphi p_\tau + \frac{\coth r \sin \varphi}{\cosh \tau} p_\varphi, \\
\mathcal{N}_3 &= -\cosh \tau \sin \varphi p_r + \coth r \sinh \tau \sin \varphi p_\tau - \frac{\coth r \cos \varphi}{\cosh \tau} p_\varphi, \\
\mathcal{L}_3 &= -\cos \varphi p_\tau + \frac{\sin \varphi}{\coth \tau} p_\varphi, \quad \mathcal{L}_2 = -\sin \varphi p_\tau - \frac{\cos \varphi}{\coth \tau} p_\varphi, \quad \mathcal{L}_1 = p_\varphi.
\end{aligned} \tag{2.11}$$

Using now equations (2.9), (2.10) and (2.11) it is easy to see the second Casimir operator \mathcal{C}_2 is related with the free Hamiltonian (2.6) by $\mathcal{C}_1 = -2R^2 \mathcal{H}_{\text{free}}$. Thus all the quantities (2.11) Poisson commute with free Hamiltonian (2.6) and are constants of the motion. From the seven integrals of the motion $\{\mathcal{H}_{\text{free}}, \mathcal{N}_i, \mathcal{L}_i\}$ only five are functionally independent, because of the relation (2.9) and constraint (2.8). Hence the geodesic motion with the Hamiltonian (2.6) turns out to be a *maximally superintegrable system*.

Let us now consider the spherically symmetric model, namely the Hamiltonian $\mathcal{H} = \mathcal{H}_{\text{free}} + \mathcal{V}(r)$, where $\mathcal{H}_{\text{free}}$ is given by equation (2.6) and $\mathcal{V}(r)$ is a potential function. It is obvious that the Hamilton–Jacobi equation $\mathcal{H} = \mathcal{E}$ for any central potential admit separation of variables in the pseudo-spherical system of coordinates (2.2) (and (2.3))¹. The pseudo-spherical system of coordinates corresponds to the subgroup chains $\text{SO}(2, 2) \supset \text{SO}(2, 1) \supset \text{SO}(2)$. Thus, the central symmetry of Hamiltonian \mathcal{H} implies the conservation law of the vector $\mathbf{L} = (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ with the scalar product (2.8), which we can interpret as Lorentzian “angular momentum”. In particular the first component of angular momentum $\mathcal{L}_1 = p_\varphi$ and Casimir invariant of algebra $\mathfrak{so}(2, 1)$:

$$L^2 = \mathcal{L}_1^2 - \mathcal{L}_2^2 - \mathcal{L}_3^2 = -\left(p_\tau^2 - \frac{p_\varphi^2}{\cosh^2 \tau}\right), \tag{2.12}$$

together with the Hamiltonian \mathcal{H} :

$$\mathcal{H} = \frac{1}{2R^2} \left\{ p_r^2 + \frac{L^2}{\sinh^2 r} \right\} + \mathcal{V}(r),$$

form the mutually Poisson-involutive system of constants of motion. As it follows from the equation (2.12): $p_\varphi^2 / \cosh^2 \tau - L^2 \geq 0$, the quantity L^2 , in contrast to the motion in Euclidean space (or spheres and two-sheeted hyperboloids), can take not only the positive or zero but also the negative value. Another difference is that at the fixed values of L^2 : $p_\varphi^2 \geq L^2$. The existence of an additional independent constant of motion \mathcal{L}_2 (\mathcal{L}_3 then not independent) means that the problem is at least once degenerate and the trajectories placed on the two-dimensional surface. For the case of positive L^2 putting $\tau = 0$, or $L^2 = p_\varphi^2$, we obtain that the motion takes place on the two-dimensional subspace, namely two-sheeted hyperboloid $z_0^2 - z_2^2 - z_3^2 = R^2$, while for negative L^2 , we may put $\varphi = 0$ or $p_\varphi^2 = 0$, and restricted to the one-sheeted hyperboloid $z_0^2 + z_1^2 - z_2^2 = R^2$.

¹Beside of the pseudo-spherical system of coordinates (2.2) the Hamilton–Jacobi equation $\mathcal{H}_{\text{free}} = \mathcal{E}$ and free Schrödinger equation on H_2^2 hyperboloid allow the separation of variables additionally in 70th orthogonal systems of coordinates (see for details [29]).

In the case of $|z_0| < R$ the formulas for $\mathfrak{so}(2, 2)$ generators (2.11) are changed accordingly to the transformation (2.4). We have

$$L^2 = - \left(p_\mu^2 + \frac{p_\varphi^2}{\sinh^2 \mu} \right).$$

Hence by virtue of above relation, the L^2 takes only negative value. Without the loss of generality we can put $\varphi = 0$ or $p_\varphi^2 = 0$ and the motion on H_2^2 again restricted to the one-sheeted hyperboloid $z_0^2 + z_1^2 - z_2^2 = R^2$.

3 Harmonic oscillator potential

Let us now concentrate on the spherically symmetric model, namely harmonic oscillator system. In the article [45] we have extended the Euclidean isotropic harmonic oscillator potential with the frequency ω to our space H_2^2 , which is given by

$$V^{\text{osc}} = \frac{\omega^2 R^2}{2} \left(\frac{z_2^2 + z_3^2 - z_1^2}{z_0^2} \right) = \begin{cases} \frac{\omega^2 R^2}{2} \tanh^2 r, & |z_0| \geq R, \\ -\frac{\omega^2 R^2}{2} \tan^2 \chi, & |z_0| \leq R. \end{cases}$$

Respectively the Hamiltonian may be expressed as follow

$$\mathcal{H}^{\text{osc}} = \frac{1}{2R^2} \left(p_r^2 + \frac{L^2}{\sinh^2 r} \right) + \frac{\omega^2 R^2}{2} \tanh^2 r \quad (3.1)$$

for $|z_0| \geq R$, and

$$\mathcal{H}^{\text{osc}} = -\frac{1}{2R^2} \left(p_\chi^2 + \frac{L^2}{\sin^2 \chi} \right) - \frac{\omega^2 R^2}{2} \tan^2 \chi \quad (3.2)$$

for $|z_0| \leq R$.

The Hamiltonian of the harmonic oscillator system, besides the angular momentum \mathbf{L} has additional integrals of motion quadratic in the momenta, which are associated with the generators $(\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3)$, the so called Demkov–Fradkin tensor [9, 12]:

$$\mathcal{D}_{ik} = \frac{1}{R^2} \mathcal{N}_i \mathcal{N}_k + \omega^2 R^2 \frac{z_i z_k}{z_0^2}, \quad \mathcal{D}_{ik} = \mathcal{D}_{ki}, \quad i, k = 1, 2, 3.$$

The components of \mathcal{D}_{ik} tensor Poisson commute with Hamiltonian of harmonic oscillator (3.1) and (3.2), but not necessarily with each other. In the pseudo-spherical coordinates the diagonal components of this tensor has the form

$$\begin{aligned} \mathcal{D}_{11} &= \frac{\mathcal{N}_1^2}{R^2} + \omega^2 R^2 \sinh^2 \tau \tanh^2 r, & \mathcal{D}_{22} &= \frac{\mathcal{N}_2^2}{R^2} + \omega^2 R^2 \cosh^2 \tau \cos^2 \varphi \tanh^2 r, \\ \mathcal{D}_{33} &= \frac{\mathcal{N}_2^2}{R^2} + \omega^2 R^2 \cosh^2 \tau \sin^2 \varphi \tanh^2 r, \end{aligned}$$

so the harmonic oscillator Hamiltonian is given by

$$\mathcal{H}^{\text{osc}} = -\mathcal{D}_{11} + \mathcal{D}_{22} + \mathcal{D}_{33} - \frac{L^2}{2R^2}. \quad (3.3)$$

In addition to this, the Demkov–Fradkin tensor has the algebraic properties

$$\sum_i \mathcal{L}_i \mathcal{D}_{ik} = \sum_i \mathcal{D}_{ki} \mathcal{L}_i = 0, \quad k = 1, 2, 3. \quad (3.4)$$

It is clear that the ten integrals of motion $\{\mathcal{H}, \mathcal{L}_i, \mathcal{D}_{ik}\}$ cannot be functionally independent because of the relations (3.3) and (3.4), and that

$$\{\mathcal{L}_1 \mathcal{D}_{11}\} = \{\mathcal{L}_2 \mathcal{D}_{22}\} = \{\mathcal{L}_3 \mathcal{D}_{33}\} = 0.$$

Only five integrals of motion, which we can choose as $\{\mathcal{H}, L^2, \mathcal{L}_1, \mathcal{L}_2, \mathcal{D}_{33}\}$, are functionally independent. Thus \mathcal{H}^{osc} is a maximally superintegrable Hamiltonian. The components of angular momentum and Demkov–Fradkin tensor forms the quadratic algebra. The nonvanishing Poisson brackets have been presented in Appendix A.

In the contraction limit $R \rightarrow \infty$ the H_2^2 hyperbolic space turns into the Minkowski space \mathbf{M}^{2+1} . Let us pass to Beltrami coordinates

$$x_i = R \frac{z_i}{z_0} = R \frac{z_i}{\sqrt{R^2 + z_2^2 + z_3^2 - z_1^2}}, \quad i = 1, 2, 3. \quad (3.5)$$

Then, at the limit $R \rightarrow \infty$ we have that

$$\lim_{R \rightarrow \infty} V^{\text{osc}}(r) = \frac{\omega^2}{2} (-x_1^2 + x_2^2 + x_3^2),$$

which can be interpreted as a harmonic oscillator potential on the \mathbf{M}^{2+1} Minkowski space (x_1, x_2, x_3) .

4 Integration of the Hamilton–Jacobi equation

The Hamilton–Jacobi equation, associated with the Hamiltonian (3.1), is obtained after the substitution $p_{\mu_i} \rightarrow \partial S / \partial \mu_i$, where $\mu_i = (r, \tau, \varphi)$. Therefore we get

$$\mathcal{H} = \frac{1}{2R^2} \left\{ \left(\frac{\partial S}{\partial r} \right)^2 - \frac{1}{\sinh^2 r} \left(\frac{\partial S}{\partial \tau} \right)^2 + \frac{1}{\sinh^2 r \cosh^2 \tau} \left(\frac{\partial S}{\partial \varphi} \right)^2 \right\} + \frac{\omega^2 R^2}{2} \tanh^2 r = E.$$

This equation is completely separable, and the coordinate φ is cyclic. We look the solution for the classical action $S(r, \tau, \varphi)$ in form

$$S(r, \tau, \varphi) = p_\varphi \varphi + S_1(r) + S_2(\tau) - Et,$$

and obtain

$$\left(\frac{\partial S_2}{\partial \tau} \right)^2 - \frac{p_\varphi^2}{\cosh^2 \tau} = -L^2, \quad (4.1)$$

$$\frac{1}{2R^2} \left(\frac{\partial S_1}{\partial r} \right)^2 + \frac{\omega^2 R^2}{2} \tanh^2 r + \frac{L^2}{2R^2 \sinh^2 r} = E. \quad (4.2)$$

The “quasi-radial” equation (4.2) describes the motion in field of effective potential

$$U_{\text{eff}}(r) = \frac{\omega^2 R^2}{2} \tanh^2 r + \frac{L^2}{2R^2 \sinh^2 r}. \quad (4.3)$$

At the large $r \sim \infty$ the effective potential $U_{\text{eff}}(r)$ tends to a constant value equal to $\omega^2 R^2 / 2$, whereas the behavior at the point $r = 0$ is determined by the angular momentum L^2 .

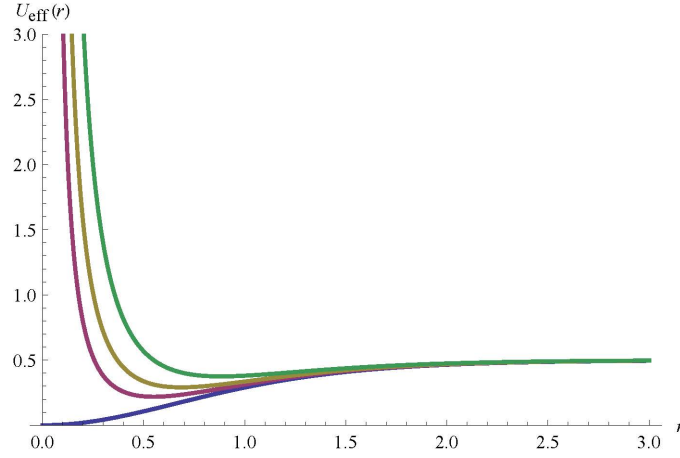


Figure 1. Effective potential $U_{\text{eff}}(r)$ in case of $0 \leq L^2 < \omega^2 R^4$ for value of $L^2 = 0, 1/16, 1/8, 1/4$; $\omega = R = 1$.

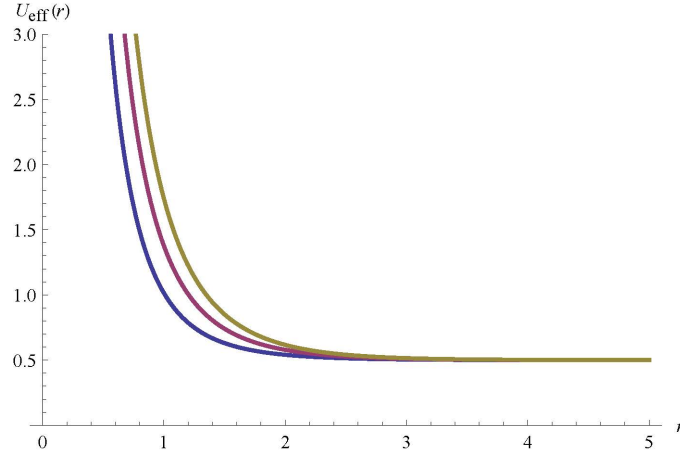


Figure 2. Effective potential $U_{\text{eff}}(r)$ in case of $L^2 \geq \omega^2 R^4$ for value of $L^2 = 2, 3, 4$; $\omega = R = 1$.

In case $0 \leq L^2 < \omega^2 R^4$ potential (4.3) has a minimum at $r_0 = \tanh^{-1} \sqrt[4]{L^2/\omega^2 R^4}$ (see Fig. 1), and at this point

$$0 \leq U_{\text{eff}}(r_0) = \omega \sqrt{L^2} - \frac{L^2}{2R^2} < \frac{\omega^2 R^2}{2}, \quad (4.4)$$

where equality is possible only in case of $L^2 = 0$. For $L^2 \geq \omega^2 R^4$ the potential $U_{\text{eff}}(r)$ is repulsive on the whole semi-axis $r \in [0, \infty)$ (see Fig. 2). In the case of negative L^2 the effective potential (4.4) is attractive and has a singularity for a small r as $\sim r^{-2}$ (see Fig. 3).

For the region $|z_0| < R$ the differential equations (4.1) and (4.2) are transformed to the following ones

$$\begin{aligned} \left(\frac{\partial S_2}{\partial \mu} \right)^2 + \frac{p_\varphi^2}{\sinh^2 \mu} &= -L^2, \\ \frac{1}{2R^2} \left(\frac{\partial S_1}{\partial \chi} \right)^2 + \frac{\omega^2 R^2}{2} \tan^2 \chi + \frac{L^2}{2R^2 \sin^2 \chi} &= -E. \end{aligned}$$

The first equation admits only negative value of L^2 . Therefore we take into account the motion inside the region $|z_0| < R$ when investigate we the case of negative value of L^2 .

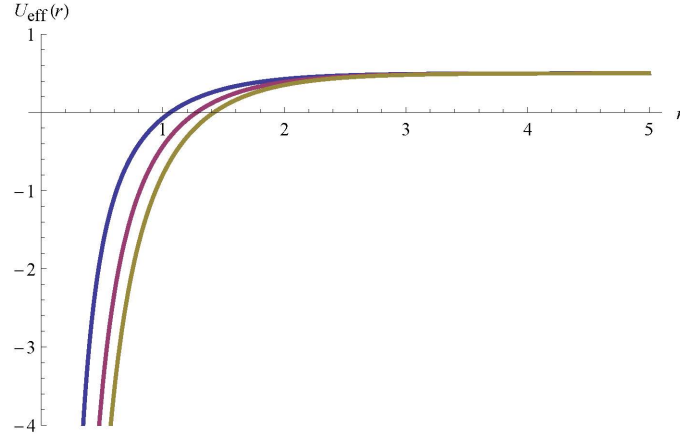


Figure 3. Effective potential $U_{\text{eff}}(r)$ in case of $L^2 < 0$ for value of $L^2 = -1, -2, -3$; $\omega = R = 1$.

Integrating now equations (4.1) and (4.2) we get

$$S_1(r) = \int \sqrt{2R^2 E - \omega^2 R^4 \tanh^2 r - \frac{L^2}{\sinh^2 r}} dr, \quad (4.5)$$

$$S_2(\tau) = \int \sqrt{-L^2 + \frac{p_\varphi^2}{\cosh^2 \tau}} d\tau. \quad (4.6)$$

Since we are interested only the trajectories we will follow the usual procedures [35] and consider the equations

$$\frac{\partial S}{\partial E} = \frac{\partial S_1}{\partial E} - t = -t_0, \quad \frac{\partial S}{\partial L^2} = \frac{\partial S_1}{\partial L^2} + \frac{\partial S_2}{\partial L^2} = \beta, \quad \frac{\partial S}{\partial p_\varphi} = \varphi + \frac{\partial S_2}{\partial p_\varphi} = \varphi_0, \quad (4.7)$$

where t_0 , φ_0 and β are the constants.

4.1 Integration of quasi-radial part

From equations (4.5) and (4.7) we get that

$$t - t_0 = \frac{1}{\omega} \int \frac{\tanh r dr}{\sqrt{-\tanh^4 r + 2(E/\omega^2 R^2 + L^2/2\omega^2 R^4) \tanh^2 r - L^2/\omega^2 R^4}}. \quad (4.8)$$

Below we consider separately all four cases: $0 < L^2 < \omega^2 R^4$, $L^2 \geq \omega^2 R^4$, $L^2 < 0$ and $L^2 = 0$.

1. The case $0 < L^2 < \omega^2 R^4$. For the roots in the radical expression of denominator in (4.8) we have

$$X_{1,2} = \frac{(2R^2 E + L^2) \pm \sqrt{(2R^2 E + L^2)^2 - 4L^2 \omega^2 R^4}}{2\omega^2 R^4}, \quad (4.9)$$

where $X = \tanh^2 r \in [0, 1]$. It's obvious that the radicand in equation (4.9) is positive for any values of energy $E > E_{\min} = U_{\text{eff}}(r_0)$ and equal zero for $E = E_{\min}$. Thus the roots $X_{1,2}$ ($X_1 \leq X_2$) are positive. It is easy to see that for $E_{\min} \leq E < \omega^2 R^2/2$ both roots satisfy the inequality condition $0 < X_1 < X_2 < 1$. At $E \geq \omega^2 R^2/2$: $0 < X_1 < 1 \leq X_2$ and equality $X_2 = 1$ is possible only for $E = \omega^2 R^2/2$. The bounded motion exists exclusively for $E_{\min} \leq E < \omega^2 R^2/2$. Below we will consider separately all possible cases, namely: $E_{\min} < E < \omega^2 R^2/2$, $E = E_{\min}$, $E > \omega^2 R^2/2$ and $E = \omega^2 R^2/2$.

A. Performing the integration in formula (4.8) we get for $E_{\min} < E < \omega^2 R^2/2$

$$2\omega^2 R^2 \sinh^2 r = (1 - 2E/\omega^2 R^2)^{-1} \left\{ (2E - L^2/R^2) + \sqrt{(2E + L^2/R^2)^2 - 4L^2\omega^2 \sin [2\omega\sqrt{1 - 2E/\omega^2 R^2}(t - t_0)]} \right\}.$$

Thus the motion is bounded and periodic. The period is given by

$$T(R) = \frac{\pi}{\omega} \frac{1}{\sqrt{1 - 2E/\omega^2 R^2}}. \quad (4.10)$$

The total frequency $\omega_0 = \omega\sqrt{1 - 2E/\omega^2 R^2}$ and unlike the motion in Euclidean space, depends on the energy of particle E and curvature of the space $\kappa = -1/R^2$ as a parameter, but it is constant for each of the orbits at a fixed value of the energy². This property is common to all closed orbits of superintegrable systems on the spaces with constant curvature. The contraction limit $R \rightarrow \infty$ give us the correct Euclidean period: $T(R)_{R \rightarrow \infty} = \frac{\pi}{\omega}$. The period of motion on H_2^2 always larger than in Euclidean space by the factor: $1/\sqrt{1 - 2E/\omega^2 R^2}$ and tends to infinity at the limit $E \rightarrow \omega^2 R^2/2$, that is the closed orbits changes to the infinite open ones.

B. In the case of minimum energy: $E = E_{\min} = U_{\text{eff}}(r_0)$ or $E_{\min} = \omega\sqrt{L^2} - L^2/2R^2$ the integral in (4.8) is not defined and we must solve directly the equation (4.2). From equation (4.2) we obtain

$$\left(\frac{\partial S_1}{\partial r} \right)^2 = - \left(\sqrt{L^2} \coth r - \sqrt{\omega^2 R^4} \tanh r \right)^2 \geq 0,$$

or $\partial S_1/\partial r = 0$ and $\tanh^2 r = \sqrt{L^2/\omega^2 R^4}$. Therefore

$$r = \tanh^{-1} \left(\sqrt{1 - \sqrt{1 - \frac{2E}{\omega^2 R^2}}} \right), \quad (4.11)$$

i.e., the trajectories are circles. Here from two values of $\sqrt{L^2}$ allowed by equation $E = U_{\text{eff}}(r_0)$, we choose the smaller one $\sqrt{L^2} = \omega R^2 \left(1 - \sqrt{1 - 2E/\omega^2 R^2} \right)$ because it satisfies the condition $0 < L^2 < \omega^2 R^4$. In case of contraction limit $R \rightarrow \infty$ we obtain $E = E_{\min} = \omega\sqrt{L^2}$ and $r = \sqrt{E}/\omega$.

C. In case of $E > \omega^2 R^2/2$ after integration in (4.8) we have

$$2\omega^2 R^2 \sinh^2 r = (2E/\omega^2 R^2 - 1)^{-1} \left\{ (L^2/R^2 - 2E) + \sqrt{(2E + L^2/R^2)^2 - 4L^2\omega^2 \cosh [2\omega\sqrt{2E/\omega^2 R^2 - 1}(t_0 - t)]} \right\}, \quad (4.12)$$

i.e., the motion is not bounded.

D. For the limiting case of $E = \omega^2 R^2/2$ the roots of denominator are $X_1 = L^2/\omega^2 R^4$, $X_2 = 1$, thus $L^2/\omega^2 R^4 < \tanh^2 r < 1$ and motion is not bounded because of $\tanh^{-1}(L^2/\omega^2 R^4) < r < \infty$. The integration in (4.8) yield

$$\cosh^2 r = (1 - L^2/\omega^2 R^4)^{-1} + \omega^2 (1 - L^2/\omega^2 R^4)(t - t_0)^2. \quad (4.13)$$

2. Let us consider now the case of $L^2 \geq \omega^2 R^4$ (see Fig. 2). From equation (4.8) we get that the only possible value for energy is $E > \omega^2 R^2/2$ and the roots satisfy the inequality

²The Euclidean harmonic oscillator is a classical example of an *isochronous* system [5]. The period of motion of Euclidean oscillator depends only from frequency and is the same for all orbit.

$0 < X_1 < 1 < X_2$. Thereby, the equation of motion is determined by the formula (4.12). The motion of particle is limited only by the point $r_{\min} = \tanh^{-1} \sqrt{X_1}$, i.e., it has the ability to go to infinity.

3. Let us consider finally the case of $L^2 \leq 0$. From the equation (4.8) we have that the roots of denominator are

$$X_{1,2} = \frac{(2ER^2 - |L^2|) \pm \sqrt{(2ER^2 - |L^2|)^2 + 4|L^2|\omega^2 R^4}}{2\omega^2 R^4},$$

where again $X = \tanh^2 r \in [0, 1]$. It can be seen that $X_1 < 0 < X_2$ is independent of the value of A and energy E . For the region $E \geq \omega^2 R^2/2$ one of the roots is $X_2 > 1$, so the radicand is positive for any values of variable r , including the point $r = 0$: $r \in [0, \infty)$. The same situation develops for region $E < \omega^2 R^2/2$, where $r \in [0, \tanh^{-1} \sqrt{X_2}]$. Therefore in case of negative A the particle can penetrate from the region $z_0 \geq R$ to $0 \leq z_0 \leq R$.

Performing the integration in formula (4.8), we have for $E < \omega^2 R^2/2$

$$\begin{aligned} \sinh^2 r = & \frac{2R^2 E + |L^2|}{2R^2(\omega^2 R^2 - 2E)} \\ & + \frac{\sqrt{(2R^2 E - |L^2|)^2 + 4|L^2|\omega^2 R^4}}{2R^2(\omega^2 R^2 - 2E)} \sin \left[2\omega \sqrt{1 - 2E/\omega^2 R^2} (t - t_0) \right], \end{aligned} \quad (4.14)$$

while for $E > \omega^2 R^2/2$

$$\begin{aligned} \sinh^2 r = & \frac{2R^2 E + |L^2|}{2R^2(\omega^2 R^2 - 2E)} \\ & + \frac{\sqrt{(2R^2 E - |L^2|)^2 + 4|L^2|\omega^2 R^4}}{2R^2(2E - \omega^2 R^2)} \cosh \left[2\omega \sqrt{2E/\omega^2 R^2 - 1} (t - t_0) \right]. \end{aligned} \quad (4.15)$$

From the formula (4.14) it follows that the motion at $E < \omega^2 R^2/2$ is bounded and periodic with period (4.10). Below we will construct the bounded trajectories lying on the whole hyperboloid, namely not only in the region $|z_0| \geq R$, but also $|z_0| \leq R$. In case when $E = \omega^2 R^2$ the integration in (4.8) leads, up to a transformation $L^2 \rightarrow -|L^2|$, to a result similar to the formula (4.13).

In the limiting case of $L^2 = 0$ the formulas (4.14), (4.15) and (4.13) are simplified. For $0 < E < \omega^2 R^2/2$ we get

$$\sinh^2 r = \frac{2E/\omega^2 R^2}{1 - 2E/\omega^2 R^2} \cos^2 \left(\omega \sqrt{1 - 2E/\omega^2 R^2} (t - t_0) - \frac{\pi}{4} \right),$$

while in case of $E > \omega^2 R^2/2$

$$\sinh r = \sqrt{\frac{2E/\omega^2 R^2}{2E/\omega^2 R^2 - 1}} \sinh \left(\omega \sqrt{2E/\omega^2 R^2 - 1} (t_0 - t) \right).$$

Finally for $E = \omega^2 R^2/2$ we obtain $\sinh r = \omega(t - t_0)$.

4.2 Integration of the angular parts

1. Let us first consider the case when $L^2 > 0$. From (4.5) and (4.6) we obtain

$$\frac{\partial S_1}{\partial L^2} = -\frac{1}{2} \int \frac{dr}{\sinh^2 r \sqrt{2R^2 E - \omega^2 R^4 \tanh^2 r - L^2 / \sinh^2 r}}, \quad (4.16)$$

$$\frac{\partial S_2}{\partial L^2} = -\frac{1}{2} \int \frac{d\tau}{\sqrt{-L^2 + p_\varphi^2 / \cosh^2 \tau}}. \quad (4.17)$$

The integrals can be easily calculated to give [15]

$$\begin{aligned} \frac{\partial S_2}{\partial L^2} &= -\frac{1}{\sqrt{4L^2}} \arcsin \left[\frac{\sinh \tau}{\sqrt{p_\varphi^2 / L^2 - 1}} \right], \\ \frac{\partial S_1}{\partial L^2} &= \frac{1}{4\sqrt{A}} \arcsin \left[\frac{2L^2 \coth^2 r - (2ER^2 + L^2)}{\sqrt{(2ER^2 + L^2)^2 - 4L^2\omega^2 R^4}} \right]. \end{aligned}$$

Here we require

$$-\sqrt{p_\varphi^2 / L^2 - 1} < \sinh \tau < \sqrt{p_\varphi^2 / L^2 - 1},$$

and

$$|2L^2 \coth^2 r - (2ER^2 + L^2)| < \sqrt{(2ER^2 + L^2)^2 - 4L^2\omega^2 R^4}. \quad (4.18)$$

The condition (4.18) is equivalent to $z_1 < \coth r < z_2$, where $z_{1,2}$ are the roots of denominator in integral (4.16):

$$z_{1,2} = \frac{(2ER^2 + L^2) \pm \sqrt{(2ER^2 + L^2)^2 - 4L^2\omega^2 R^4}}{2L^2}, \quad E \geq E_{\min} = \omega\sqrt{L^2} - L^2/2R^2.$$

The final condition $z_2 > 1$ implies that $L^2 > \omega^2 R^4$ and $E > \omega^2 R^2/2$ or $0 < L^2 < \omega^2 R^4$ and $E > E_{\min}$.

Therefore for $\partial S/\partial L^2$ we have

$$\begin{aligned} \frac{\partial S}{\partial L^2} &= \frac{1}{4\sqrt{L^2}} \left\{ \arcsin \left[\frac{2L^2 \coth^2 r - (2ER^2 + L^2)}{\sqrt{(2ER^2 + L^2)^2 - 4L^2\omega^2 R^4}} \right] \right. \\ &\quad \left. - 2 \arcsin \left[\frac{\sinh \tau}{\sqrt{p_\varphi^2 / L^2 - 1}} \right] \right\} = \beta. \end{aligned} \quad (4.19)$$

Next, from (4.6) and (4.7) we obtain

$$\frac{\partial S}{\partial p_\varphi} = \varphi + \int \frac{p_\varphi d\tau}{\cosh^2 \tau \sqrt{-L^2 + p_\varphi^2 / \cosh^2 \tau}} = \varphi + \arcsin \frac{\tanh \tau}{\sqrt{1 - L^2/p_\varphi^2}} = \varphi_0, \quad (4.20)$$

and hence

$$\tanh \tau = \sqrt{1 - L^2/p_\varphi^2} \sin(\varphi_0 - \varphi). \quad (4.21)$$

2. Let us consider the integration in formulas (4.16), (4.17) and (4.20) in the case $L^2 \leq 0$. Instead of equation (4.19) we obtain [15]

$$\frac{\partial S}{\partial L^2} = \frac{1}{4\sqrt{|L^2|}} \left\{ \operatorname{arccosh} \left[\frac{2|L^2| \coth^2 r + (2ER^2 - |A|)}{\sqrt{(2ER^2 - |L^2|)^2 + 4|L^2|\omega^2 R^4}} \right] \right.$$

$$-2 \operatorname{arcsinh} \left[\frac{\sinh \tau}{\sqrt{1 + p_\varphi^2 / |L^2|}} \right] \Bigg\} = \beta, \quad (4.22)$$

and

$$\sin(\varphi_0 - \varphi) = \frac{p_\varphi}{\sqrt{p_\varphi^2 + |L^2|}} \tanh \tau, \quad (4.23)$$

with the restriction for r :

$$\coth^2 r \geq \left(\frac{1}{2} - \frac{ER^2}{|L^2|} \right) + \sqrt{\left(\frac{1}{2} - \frac{ER^2}{|L^2|} \right)^2 + \frac{\omega^2 R^4}{|L^2|}}.$$

The limiting case of $L^2 = 0$ could be easily calculated directly from equations (4.22) and (4.23). So, we get

$$\left. \frac{\partial S}{\partial L^2} \right|_{L^2=0} = \frac{\sqrt{2E \coth^2 r - \omega^2 R^2}}{4ER} - \frac{\sinh \tau}{2p_\varphi} = \beta, \quad \sinh \tau = \tan(\varphi_0 - \varphi) \quad (4.24)$$

with the obvious restriction $\coth^2 r \geq \omega^2 R^2 / 2E$.

5 The trajectories for $L^2 > 0$

From (4.19) and (4.21) we have

$$\coth^2 r = \left(\frac{ER^2}{L^2} + \frac{1}{2} \right) + \sqrt{\left(\frac{ER^2}{L^2} + \frac{1}{2} \right)^2 - \frac{\omega^2 R^4}{L^2}} \sin(2\psi + 4\sqrt{L^2}\beta), \quad (5.1)$$

where

$$\psi = \arcsin \left(\frac{\sinh \tau}{\sqrt{p_\varphi^2 / L^2 - 1}} \right) = \arcsin \left(\frac{1}{\sqrt{1 + L^2 / p_\varphi^2 \cot^2(\varphi_0 - \varphi)}} \right). \quad (5.2)$$

Now we can rewrite the equation (5.1) in form

$$\tanh^2 r = \frac{1}{\left(\frac{ER^2}{L^2} + \frac{1}{2} \right) + \sqrt{\left(\frac{ER^2}{L^2} + \frac{1}{2} \right)^2 - \frac{\omega^2 R^4}{L^2}} \sin(2\psi + 4\sqrt{L^2}\beta)}. \quad (5.3)$$

Thus we see from (5.2) that the dependence of angle τ in the equation of trajectories (5.3) can be eliminated. On the other hand from the formula (4.21) it follows that the motion of particle on the hyperboloid is restricted to the additional condition

$$\frac{z_1}{z_3} = \frac{\tanh \tau}{\sin \varphi} = \sqrt{1 - L^2 / p_\varphi^2}.$$

Therefore, without the loss of generality we can choose $\tau = 0$ or $L^2 = p_\varphi^2$. Taking into account that the formula (5.3) is invariant about transformation $r \rightarrow i\pi - r$ we can conclude that all trajectories of motion, given by this formula, lie on the upper ($z_0 \geq R$) or lower ($z_0 \leq -R$) sheets of the two-sheeted hyperboloid: $z_0^2 - z_2^2 - z_3^2 = R^2$. Obviously they are symmetric with respect to transformation $z_0 \rightarrow -z_0$.

Putting now $L^2 = p_\varphi^2$ in (4.21) we obtain that $\psi = (\varphi_0 - \varphi)$ and the formula (5.3) gain the following form (equation of orbits)

$$\tanh^2 r = \frac{p}{1 + \varepsilon(R) \cos 2\varphi}, \quad (5.4)$$

where we use the notations

$$p(R) = \left(\frac{ER^2}{L^2} + \frac{1}{2} \right)^{-1} > 0, \quad \varepsilon(R) = \sqrt{1 - \frac{4\omega^2 R^4 L^2}{(2ER^2 + L^2)^2}} < 1, \quad (5.5)$$

and choose $\varphi_0 = -2\beta\sqrt{A} + \frac{\pi}{4}$ that the points $\varphi = 0$ will be the nearest to the center. It is clear that radicand is always positive because of $E > U_{\text{eff}}(r_0)$ for $0 < A < \omega^2 R^4$ and $E > \omega^2 R^2/2$ for $A \geq \omega^2 R^4$.

It is well-known that as in the Euclidean plane it is possible to introduce the conic (section) on the two-dimensional spaces of constant curvature [7, 10, 34] (see also the definition of curves on the two dimensional hyperboloid in [41]). The conics on the spaces with constant curvatures are the curves of the intersection between two-sheeted hyperboloid (or sphere) and second order quadric cone with the origin in the center of hyperboloid (sphere). Geometrically the conic on the spaces of constant curvature possesses many properties characteristic of conic section in Euclidean plane, particularly we can speak about the focuses F_1 and F_2 and can determine the conic as the point set, from which the sum (ellipses) or difference (hyperbolas) $2a$ of distances r_1 and r_2 to two given points (focuses F_1 and F_2) are constant.

Let us now analysis of the oscillator orbit (5.4). The formula of trajectories (5.4) may be written in more convenient form

$$\frac{1}{\tanh^2 r} = \frac{\cos^2 \varphi}{B^2} + \frac{\sin^2 \varphi}{A^2}, \quad (5.6)$$

or in term of the Beltrami coordinate (3.5):

$$\frac{x_2^2}{B^2} + \frac{x_3^2}{A^2} = R^2, \quad (5.7)$$

where the constant A and B are

$$B^2 = \frac{p(R)}{1 + \varepsilon(R)}, \quad A^2 = \frac{p(R)}{1 - \varepsilon(R)}, \quad 0 < B^2 \leq A^2. \quad (5.8)$$

The orbit equation of the type (5.6) has been studied in detail in the paper [6] (see also [10]) at the investigation of two-dimensional harmonic oscillator in the space of constant curvature in polar coordinates. The curves (5.6) are always conic on the hyperbolic plane, but its type depends on the value of A and B . It is obvious that if the value $A^2 > 1$ and $B^2 > 1$, then for any polar angle φ it follows that $\tanh r > 1$, and this case cannot produce any oscillator orbit. In the case of $B^2 < A^2 < 1$ the conic (5.6) takes the form of hyperbolic ellipses. The quantities A and B are related to the lengths of the large and small semiaxes a and b , running the interval $[0, \infty)$, defined as the values of r at $\varphi = \pi/2$ and $\varphi = 0$. Then the values A , B can be written in term of hyperbolic tangent of a , b : $A^2 = \tanh^2 a$ and $B^2 = \tanh^2 b$ and the equation of orbit (5.6) is

$$\frac{1}{\tanh^2 r} = \frac{\cos^2 \varphi}{\tanh^2 b} + \frac{\sin^2 \varphi}{\tanh^2 a}. \quad (5.9)$$

In the contraction limit $R \rightarrow \infty$ we have $r \rightarrow \tilde{r}/R$ where $\tilde{r} = \sqrt{x_2^2 + x_3^2}$ is the radial variable in the Euclidean plane. Taking into account the limit

$$\varepsilon(R) \rightarrow \tilde{\varepsilon} = \sqrt{1 - \frac{\omega^2 L^2}{E^2}}, \quad R^2 p(R) \rightarrow \tilde{p} \equiv \frac{L^2}{E},$$

we get from (5.7) that the equation of trajectories transforms into the oscillator one

$$\frac{x_2^2}{\tilde{B}^2} + \frac{x_3^2}{\tilde{A}^2} = 1, \quad \tilde{B}^2 = \frac{\tilde{p}}{1 + \tilde{\varepsilon}}, \quad \tilde{A}^2 = \frac{\tilde{p}}{1 - \tilde{\varepsilon}}$$

The next interesting case is when $B^2 < 1 < A^2$. This conic (5.6) is neither the ellipse nor the hyperbola. Following the paper [6] we will call this conic as the *ultraellipse*. Only one semi-axis b belongs to the hyperbolic plane and the next one formally is not on the real distance. It is possible to introduce a new “semiaxis” \tilde{a} (situated on the complex plane on the line $\tilde{a} = a + i\pi/2$) which related with the quantity A by $A^2 = \coth \tilde{a}$. Thus, instead of (5.9) we have the conic

$$\frac{1}{\tanh^2 r} = \frac{\cos^2 \varphi}{\tanh^2 b} + \tanh^2 \tilde{a} \sin^2 \varphi. \quad (5.10)$$

There is a joint point of two conics (5.9) and (5.10), namely $A^2 = 1$ ($a \rightarrow \infty$ or $\tilde{a} \rightarrow \infty$). In this case the conic is given by

$$\frac{1}{\tanh^2 r} = \frac{\cos^2 \varphi}{\tanh^2 b} + \sin^2 \varphi.$$

This conic is an *equidistant curve* with equidistance b from the axis z_2 [6].

Let us now consider all the possible trajectories of motion depending on the energy and angular momentum L^2 .

A. First we consider the case when $U_{\text{eff}}(r_0) < E < \omega^2 R^2/2$ and $0 < L^2 < \omega^2 R^4$. It is clear that

$$B^2 \leq A^2 = \left\{ \left(\frac{ER^2}{L^2} + \frac{1}{2} \right) - \sqrt{\left(\frac{ER^2}{L^2} + \frac{1}{2} \right)^2 - \frac{\omega^2 R^4}{L^2}} \right\}^{-1} < 1$$

and the oscillator orbits are described by the equation (5.9). Denote the minimum $b = r_{\min}$, ($\varphi = 0$) and maximum $a = r_{\max}$, ($\varphi = \pi/2$) points on the orbit as a distance from the center of field. From (5.8) and (5.9) we have

$$\tanh^2 r_{\min} = \frac{p}{1 + \varepsilon(R)}, \quad \tanh^2 r_{\max} = \frac{p}{1 - \varepsilon(R)},$$

and correspondingly

$$r_{\min} = \coth^{-1} \left\{ \sqrt{\left(\frac{ER^2}{L^2} + \frac{1}{2} \right) + \sqrt{\left(\frac{ER^2}{L^2} + \frac{1}{2} \right)^2 - \frac{\omega^2 R^4}{L^2}}} \right\},$$

$$r_{\max} = \coth^{-1} \left\{ \sqrt{\left(\frac{ER^2}{L^2} + \frac{1}{2} \right) - \sqrt{\left(\frac{ER^2}{L^2} + \frac{1}{2} \right)^2 - \frac{\omega^2 R^4}{L^2}}} \right\}.$$

Thus we find that the trajectories of motion are *ellipses* lying symmetrically to the point $z_0 = R$, $z_1 = z_2 = z_3 = 0$ on the upper sheet of the two-sheeted hyperboloid (see Fig. 4).

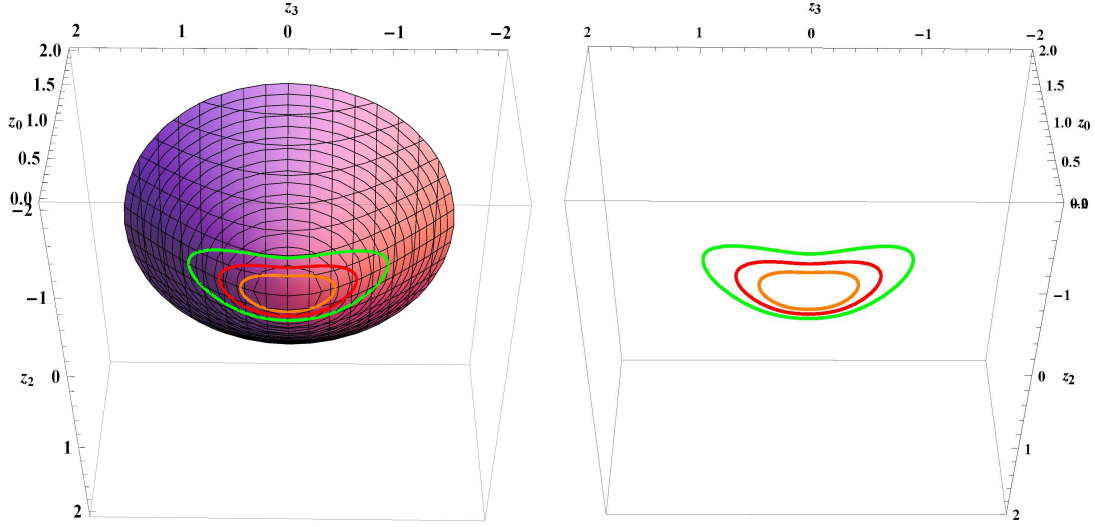


Figure 4. The figure shows the elliptic trajectories lying on the upper sheet of the two-sheeted hyperboloid $z_0^2 - z_2^2 - z_3^2 = R^2$, $z_0 > R$ for the value $\varepsilon = 0.3$ and $p = 0.3, 0.4, 0.5$.

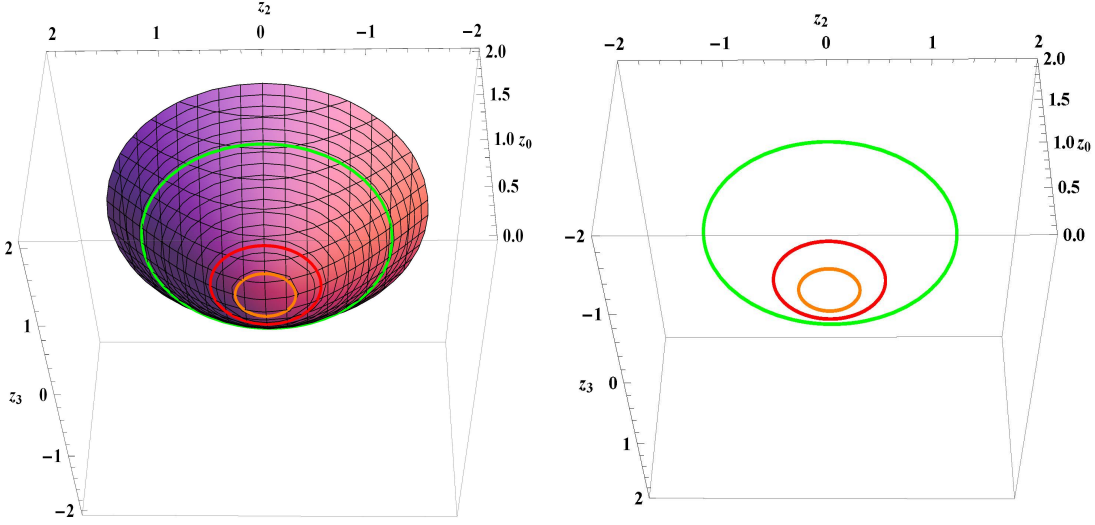


Figure 5. The cyclic orbits: $\varepsilon = 0$ and $p = 0.2, 0.5, 0.8$.

B. In case of minimum energy $E = E_{\min} = U_{\text{eff}}(r_0)$ we have from (5.5) that $\varepsilon = 0$ and $p = \omega R^2 / \sqrt{L^2}$ and consequently $\tanh^2 r = B^2 = A^2 = \omega R^2 / \sqrt{L^2}$. Thus the orbits are circles with the radius given by the formula (4.11) (see Fig. 5).

C. For the case of energy values $E = \omega^2 R^2 / 2$ we get that

$$p(R) = \frac{2A}{\omega^2 R^4 + L^2}, \quad \varepsilon(R) = \frac{|\omega^2 R^4 - L^2|}{\omega^2 R^4 + L^2},$$

therefore for $0 < L^2 < \omega^2 R^4$ we get $B^2 = L^2 / \omega^2 R^4 < 1$ and $A^2 = 1$. The conic is

$$\frac{1}{\tanh^2 r} = \frac{\omega^2 R^4}{L^2} \cos^2 \varphi + \sin^2 \varphi,$$

which represents the equidistant curves (see Fig. 6). The minimal distance r_{\min} from the center

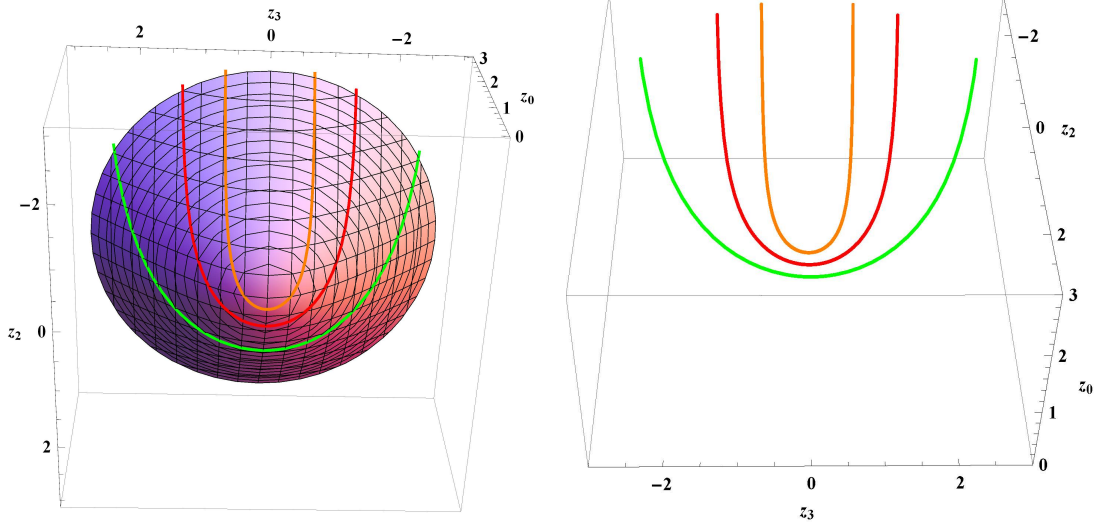


Figure 6. The figure shows the equidistant orbits lying on the upper sheet of the two-sheeted hyperboloid $z_0^2 - z_2^2 - z_3^2 = R^2$, $z_0 > R$ with the value of pairs $(p, \varepsilon) = (1/3, 2/3); (2/3, 1/3); (8/9, 1/9)$.

is given by the formula

$$r_{\min} = \coth^{-1} \left(\frac{\omega R^2}{\sqrt{L^2}} \right).$$

Let $L^2 = \omega^2 R^4$. Then $B^2 = A^2 = 1$ and the conic is a “largest” circle with radius $r = \infty$. For the case $L^2 > \omega^2 R^4$ we obtain that $B^2 = 1$, $A^2 = L^2/\omega^2 R^4 > 1$. Then from the formula (5.6) it follows that $\tanh r > 1$ and no any oscillator orbits exist.

D. For the energy $E > \omega^2 R^2/2$ it is easy to see that for any positive $L^2 > 0$

$$A^2 = \left\{ \left(\frac{ER^2}{L^2} + \frac{1}{2} \right) - \sqrt{\left(\frac{ER^2}{L^2} + \frac{1}{2} \right)^2 - \frac{\omega^2 R^4}{L^2}} \right\}^{-1} > 1, \quad B^2 < 1.$$

The motion of a particle is determined by the equation (5.10) where $\tanh^2 \tilde{a} = 1/A^2$. The trajectories are ultraellipses and describe the motion of a particle from the minimum point r_{\min} :

$$r_{\min} = \coth^{-1} \left\{ \sqrt{\left(\frac{ER^2}{L^2} + \frac{1}{2} \right)} + \sqrt{\left(\frac{ER^2}{L^2} + \frac{1}{2} \right)^2 - \frac{\omega^2 R^4}{L^2}} \right\},$$

to infinity (see Fig. 7). On the other hand side $B^2 \cdot A^2 = \omega^2 R^4/L^2$, so that for $L^2 < \omega^2 R^4$ we get $1/A^2 < B^2 < 1$, whereas for $L^2 > \omega^2 R^4$ follows that $B^2 < 1/A^2 < 1$ and the value of $L^2 = \omega^2 R^4$ or $B^2 = 1/A^2$ separates two set of ultraellipses.

Let us also note that the in contraction limit $R \rightarrow \infty$ these orbits corresponds to the Euclidean oscillator orbits with the large values of energy (the straight line $x_2^2 = \tilde{B}^2$).

6 The trajectories for $L^2 \leq 0$

To simplify further formulas we set first $p_\varphi = 0$. Then, from equation (4.23) it follows that the motion occurs at a constant value of the azimuthal angle $\varphi = \varphi_0$ that is limited by the condition $z_3/z_2 = \tan \varphi_0$. To further simplify it is enough to choose $\varphi_0 = 0$ or $\varphi_0 = \pi$. Thus we

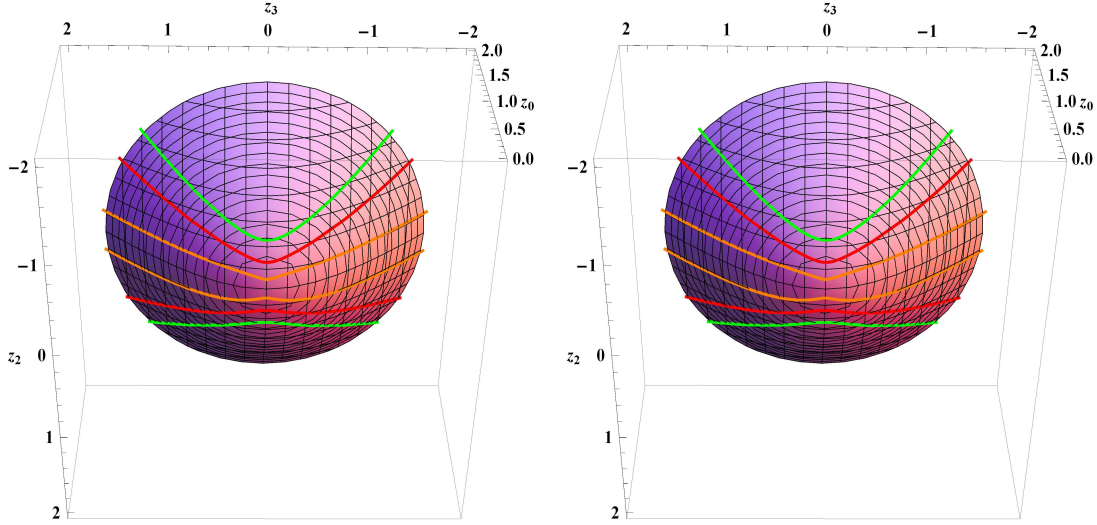


Figure 7. The figure shows the ultraellipses lying on the upper sheet of the two-sheeted hyperboloid $z_0^2 - z_1^2 - z_2^2 = R^2$, $z_0 > R$ for the value $\epsilon = 0.8$ and $p = 0.2, 0.5, 0.8$.

get that trajectory of the motion lies on the one-sheeted hyperboloid $z_0^2 + z_1^2 - z_2^2 = R^2$. The formula (4.22) gives us the equation of the trajectory in the region $z_0 > R$:

$$\coth^2 r = \left(\frac{1}{2} - \frac{ER^2}{|L^2|} \right) + \sqrt{\left(\frac{1}{2} - \frac{ER^2}{|L^2|} \right)^2 + \frac{\omega^2 R^4}{|L^2|}} \cosh(2\tau + 4\sqrt{|L^2|}\beta). \quad (6.1)$$

Performing the further transformation $r \rightarrow i\chi$ and $\tau \rightarrow \mu - i\pi/2$ in formula (6.1), we obtain the equation of the trajectory in the region $0 < z_0 < R$:

$$\cot^2 \chi = -\left(\frac{1}{2} - \frac{ER^2}{|L^2|} \right) + \sqrt{\left(\frac{1}{2} - \frac{ER^2}{|L^2|} \right)^2 + \frac{\omega^2 R^4}{|L^2|}} \cosh(2\mu + 4\sqrt{|L^2|}\beta). \quad (6.2)$$

In the formula of trajectory (6.1) we must distinguish two cases, namely for the value of energy $E < \omega^2 R^2/2$ and $E \geq \omega^2 R^2/2$.

In the first case $E < \omega^2 R^2/2$ from equation (6.1) it follows that for any value of the variable $\tau \in (-\infty, \infty)$ we have that $\coth r > 1$. Therefore, the trajectory of the motion extends from the point $r = 0$ at the $\tau \rightarrow -\infty$ ($z_0 = R$, $z_1 < 0$, $z_2 > 0$) to its maximum

$$r_{\max} = \coth^{-1} \sqrt{\left(\frac{1}{2} - \frac{ER^2}{|L^2|} \right) + \sqrt{\left(\frac{1}{2} - \frac{ER^2}{|L^2|} \right)^2 + \frac{\omega^2 R^4}{|L^2|}}},$$

at the point $\tau = -2\sqrt{|L^2|}\beta$ and then goes back to the point $r = 0$ when $\tau \rightarrow \infty$ ($z_0 = R$, $z_1 > 0$, $z_2 > 0$). Further on, the particle penetrates through the point $z_0 = R$ from the region $z_0 > R$ to the region $0 < z_0 < R$, which, as it follows from the equation (6.2), corresponds to the value of angles $\mu \rightarrow \infty$ and $\chi \rightarrow 0$, ($z_0 < R$, $z_1 > 0$, $z_2 > 0$). Further trajectory extends to the maximal value χ_{\max} :

$$\chi_{\max} = \cot^{-1} \sqrt{-\left(\frac{1}{2} - \frac{ER^2}{|L^2|} \right) + \sqrt{\left(\frac{1}{2} - \frac{ER^2}{|L^2|} \right)^2 + \frac{\omega^2 R^4}{|L^2|}}} \leq \frac{\pi}{2},$$

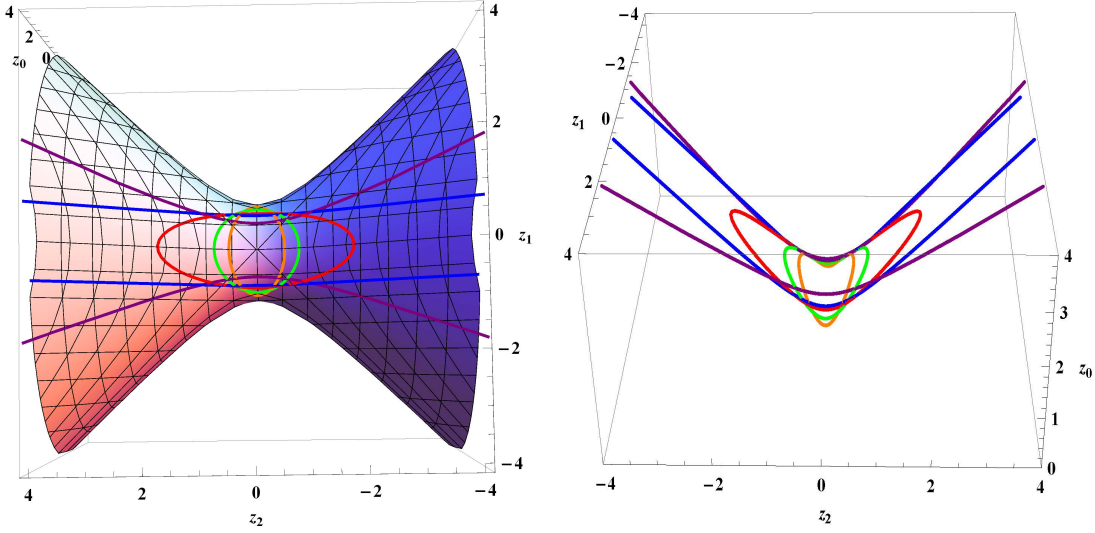


Figure 8. The trajectories of motion in the case of $|L^2| = 1$; $E = -3/2, -1/2, 1/4, 1/2, 3/2$; $\omega = R = 1$.

at the point $\mu = -2\sqrt{|L^2|}\beta$, and then continue to $\mu \rightarrow -\infty$, $\chi \rightarrow 0$ ($z_0 < R$, $z_1 > 0$, $z_2 < 0$). After, the particle again passes the point $z_0 = R$ and penetrates to the region $z_0 \geq R$. Further using similar reasoning it can be shown that the trajectories in case of $E < \omega^2 R^2/2$, are a closed curve lying on the one-sheeted hyperboloid $z_0^2 + z_1^2 - z_2^2 = R^2$, $z_0 > 0$, so the motions are bounded and periodic. The same situation takes place for the case of $z_0 < 0$.

In the case of $E \geq \omega^2 R^2/2$ it is easy to see that the inequality

$$\sqrt{\left(\frac{1}{2} - \frac{ER^2}{|A|}\right)^2 + \frac{\omega^2 R^4}{|L^2|}} \leq \frac{1}{2} + \frac{ER^2}{|L^2|}$$

is valid. Thus the trajectory of the motion, depending on the sign of variable τ is split into two paths. One of the paths begins from the large r at the minimal point

$$\tau_{\min} = -2\sqrt{|L^2|}\beta + \frac{1}{2} \cosh^{-1} \frac{\left(\frac{1}{2} + \frac{ER^2}{|L^2|}\right)}{\sqrt{\left(\frac{1}{2} - \frac{ER^2}{|L^2|}\right)^2 + \frac{\omega^2 R^4}{|L^2|}}}.$$

and continues to the point $r = 0$ at $\tau \rightarrow \infty$ ($z_0 = R$, $z_2 > 0$). Then the trajectory passing the part of $0 < z_0 < R$ goes back from ($z_0 = R$, $z_2 < 0$) at the point $r = 0$, $\tau \sim \infty$ to $r \in \infty$ at τ_{\min} . The second path is symmetric with respect to axis z_1 . Thus the trajectories of motion in the case of $E \geq \omega^2 R^2/2$ are not bounded. Some examples of trajectories for the fixed negative L^2 and various values of energy E , are presented on the Fig. 8.

In the case $L^2 = 0$ it is easy to get from (4.24)

$$\coth^2 r = \frac{\omega^2 R^2}{2E} + R\sqrt{E} (2\beta - \tan \varphi/p_\varphi)^2,$$

with $\varphi_0 = 0$. In the case of $E < \omega^2 R^2/2$ the bounded motion takes place $r_{\min} = 0$ ($\varphi = \pi/2$) and $r_{\max} = \coth^{-1} \sqrt{\frac{\omega^2 R^2}{2E}}$ ($\varphi = \arctan 2\beta p_\varphi$), whereas for the $E \geq \omega^2 R^2/2$ the orbits are infinite: $r \in [0, \infty)$. The trajectories of the motion can be presented on the hyperbolic cylinder $z_0^2 - z_2^2 = R^2$, $z_1^2 = z_3^2$, $z_0 \geq R$ (see Fig. 9).

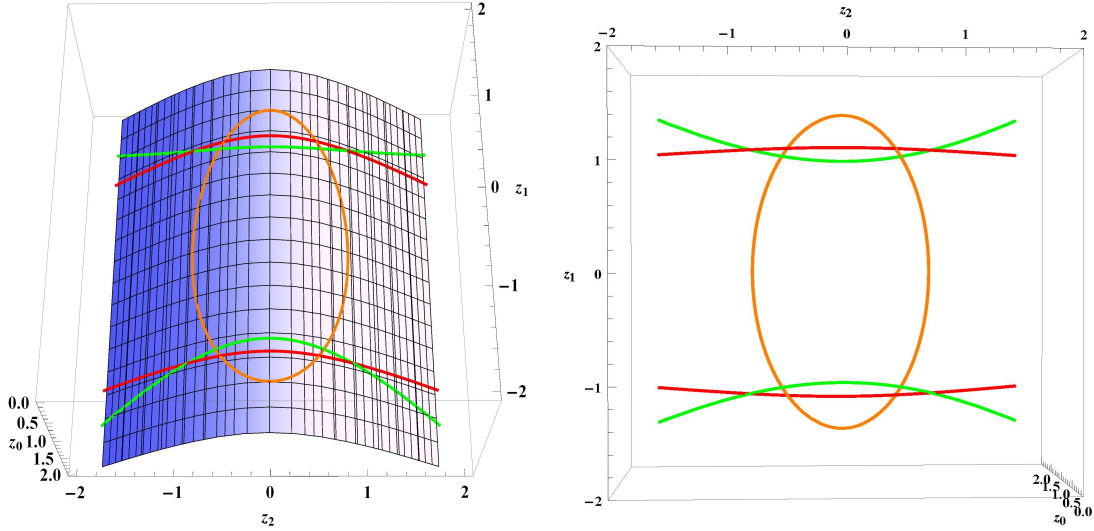


Figure 9. The bounded and infinite trajectories of the motion for $L^2 = 0$ lying on the hyperbolic cylinder $z_0^2 - z_2^2 = R^2$, $z_1^2 = z_3^2$, and $z_0 \geq R$. The figure shows the cases $E = 0.2, 0.5, 0.8$; $\omega = R = p_\varphi = 1$.

7 Conclusion

We have shown that the notion of harmonic oscillator problem can be extended not only to the sphere and two-sheeted hyperboloid but also to the hyperbolic space H_2^2 . It was proved that the harmonic oscillator problem on H_2^2 is exactly solvable and also belongs to the class of superintegrable systems. We have constructed the dynamical algebra of symmetry for this system, which is nonlinear and quadratic (so-called Higgs algebra). We completely solved the Hamilton–Jacobi equation for harmonic oscillator problem in the geodesic pseudo-spherical systems of coordinates. It was shown that for positive value of the Lorentzian momentum $L^2 > 0$ all trajectories of motion lie on the upper (or lower) sheets of two dimensional two-sheeted hyperboloid $z_0^2 - z_2^2 - z_3^2 = R^2$. These trajectories are always conics centered in the origin of potential $r = 0$. For the special values of energy $E_{\min} < E < \omega^2 R^2/2$ and momentum $L^2 < \omega^2 R^4$ all the orbits are ellipses (or circles for $E = E_{\min}$). In case when $E > \omega^2 R^2/2$ independently of the value of L^2 , the oscillator orbits are ultraellipses or equidistant curves for $E = \omega^2 R^2/2$. We have seen that in case of negative values of Lorentzian momentum $L^2 \leq 0$ the oscillator orbits lie on the one-sheeted hyperboloid $z_0^2 + z_1^2 - z_2^2 = R^2$ and are bounded and periodic for $E < \omega^2 R^2/2$ and infinite for $E \geq \omega^2 R^2/2$. The similar situation is valid for $L^2 = 0$, but in this case the orbits lie on the hyperbolic cylinder $z_0^2 - z_2^2 = R^2$, $z_1^2 = z_3^2$.

Let us make short comments concerning the connection of the classical and quantum case. The quantum-mechanical counterpart of the angular momentum operator (2.7) comes through the replacement $p_\mu \rightarrow -i\partial/\partial z_\mu$ and is given by

$$\hat{L}_1 = -i(z_2\partial_3 - z_3\partial_2), \quad \hat{L}_2 = -i(z_1\partial_3 + z_3\partial_1), \quad \hat{L}_3 = i(z_1\partial_2 + z_2\partial_1).$$

Then in the pseudo-spherical coordinates (2.2) the operator \hat{L}^2 takes the form

$$\hat{L}^2 = \hat{L}_1^2 - \hat{L}_2^2 - \hat{L}_3^2 = \left(\frac{1}{\cosh \tau} \frac{\partial}{\partial \tau} \cosh \tau \frac{\partial}{\partial \tau} - \frac{1}{\cosh^2 \tau} \frac{\partial^2}{\partial \varphi^2} \right),$$

and coincide with the Casimir operator of $SO(2,1)$ group. Thus the Schrödinger equation for the harmonic oscillator potential can be written as

$$\frac{1}{\sinh^2 r} \frac{\partial}{\partial r} \sinh^2 r \frac{\partial \Psi}{\partial r} + \left[2R^2 E - \frac{\hat{L}^2}{\sinh^2 r} - \omega^2 R^4 \tanh^2 r \right] \Psi = 0, \quad (7.1)$$

and solved by separation of variables via the ansatz $\Psi(r, \tau, \varphi) = \mathcal{R}(r)\mathcal{Y}(\tau, \varphi)$. The pseudo-spherical function \mathcal{Y} is a eigenfunction of operator $\hat{L}^2\mathcal{Y} = \ell(\ell+1)\mathcal{Y}$ which describes the quantum geodesic motion on the two-dimensional one-sheeted hyperboloid. The spectrum of ℓ can take as well as the real values: $\ell = 0, 1, \dots$ (discrete series of representation of SO(2, 1) group) and complex value $\ell = -1/2 + i\rho$, $\rho > 0$ (continuous principal series). In the first case the eigenvalue of \hat{L}^2 operator is positive and in the second one negative. The exact solution of the Schrödinger equation (7.1) for the positive eigenvalues of operator \hat{L}^2 has been constructed in the previous paper [45]. It was shown that as in the case of two-sheeted hyperboloid, the energy spectrum contains the scattering states and a finite number of degenerate bound states. This fact coincides with the existence of closed and infinite orbits for positive L^2 in classical case. We have not considered in the article [45] the quantum motion in the case of negative eigenvalue of \hat{L}^2 because of the strong singularity at the center of harmonic oscillator potential, although it is clear that the system has a discrete spectrum. This work is in progress.

Finally, we wish to emphasize that the Kepler–Coulomb and harmonic oscillator potentials are the “building block” upon which most of superintegrable potentials can be constructed. Thus the investigation of these systems is important for the further study and understanding of more complicated superintegrable systems in the hyperbolic space H_2^2 .

A Symmetry algebra

The nonvanishing Poisson brackets between the components of Demkov–Fradkin tensor \mathcal{D}_{ij} and \mathcal{L}_i :

$$\begin{aligned} \{\mathcal{D}_{12}, \mathcal{L}_1\} &= -\mathcal{D}_{13}, & \{\mathcal{D}_{12}, \mathcal{L}_2\} &= -\mathcal{D}_{23}, & \{\mathcal{D}_{12}, \mathcal{L}_3\} &= -\mathcal{D}_{11} - \mathcal{D}_{22}, \\ \{\mathcal{D}_{13}, \mathcal{L}_1\} &= \mathcal{D}_{12}, & \{\mathcal{D}_{13}, \mathcal{L}_2\} &= -\mathcal{D}_{11} - \mathcal{D}_{33}, & \{\mathcal{D}_{13}, \mathcal{L}_3\} &= -\mathcal{D}_{23}, \\ \{\mathcal{D}_{23}, \mathcal{L}_1\} &= \mathcal{D}_{22} - \mathcal{D}_{33}, & \{\mathcal{D}_{23}, \mathcal{L}_2\} &= -\mathcal{D}_{12}, & \{\mathcal{D}_{23}, \mathcal{L}_3\} &= -\mathcal{D}_{13}, \\ \{\mathcal{D}_{11}, \mathcal{L}_2\} &= -2\mathcal{D}_{13}, & \{\mathcal{D}_{11}, \mathcal{L}_3\} &= -2\mathcal{D}_{12}, & \{\mathcal{D}_{22}, \mathcal{L}_1\} &= -2\mathcal{D}_{23}, \\ \{\mathcal{D}_{22}, \mathcal{L}_3\} &= -2\mathcal{D}_{12}, & \{\mathcal{D}_{33}, \mathcal{L}_1\} &= 2\mathcal{D}_{23}, & \{\mathcal{D}_{33}, \mathcal{L}_2\} &= -2\mathcal{D}_{13}, \end{aligned}$$

The same between \mathcal{D}_{ik} :

$$\begin{aligned} \{\mathcal{D}_{11}, \mathcal{D}_{12}\} &= 2\omega^2\mathcal{L}_3 + \frac{2}{R^2}\mathcal{L}_3\mathcal{D}_{11}, & \{\mathcal{D}_{11}, \mathcal{D}_{13}\} &= 2\omega^2\mathcal{L}_2 + \frac{2}{R^2}\mathcal{L}_2\mathcal{D}_{11}, \\ \{\mathcal{D}_{11}, \mathcal{D}_{23}\} &= \frac{2}{R^2}(\mathcal{L}_2\mathcal{D}_{12} + \mathcal{L}_3\mathcal{D}_{13}), & \{\mathcal{D}_{11}, \mathcal{D}_{22}\} &= \frac{4}{R^2}\mathcal{L}_3\mathcal{D}_{12}, \\ \{\mathcal{D}_{22}, \mathcal{D}_{12}\} &= 2\omega^2\mathcal{L}_3 - \frac{2}{R^2}\mathcal{L}_3\mathcal{D}_{22}, & \{\mathcal{D}_{22}, \mathcal{D}_{13}\} &= -\frac{2}{R^2}(\mathcal{L}_3\mathcal{D}_{23} + \mathcal{L}_1\mathcal{D}_{12}), \\ \{\mathcal{D}_{22}, \mathcal{D}_{23}\} &= 2\omega^2\mathcal{L}_1 - \frac{2}{R^2}\mathcal{L}_1\mathcal{D}_{22}, & \{\mathcal{D}_{22}, \mathcal{D}_{33}\} &= -\frac{4}{R^2}\mathcal{L}_1\mathcal{D}_{23}, \\ \{\mathcal{D}_{33}, \mathcal{D}_{12}\} &= -\frac{2}{R^2}(\mathcal{L}_2\mathcal{D}_{23} - \mathcal{L}_1\mathcal{D}_{13}), & \{\mathcal{D}_{33}, \mathcal{D}_{13}\} &= 2\omega^2\mathcal{L}_2 - \frac{2}{R^2}\mathcal{L}_2\mathcal{D}_{33}, \\ \{\mathcal{D}_{33}, \mathcal{D}_{23}\} &= -2\omega^2\mathcal{L}_1 + \frac{2}{R^2}\mathcal{L}_1\mathcal{D}_{33}, & \{\mathcal{D}_{33}, \mathcal{D}_{11}\} &= -\frac{4}{R^2}\mathcal{L}_2\mathcal{D}_{13}, \\ \{\mathcal{D}_{12}, \mathcal{D}_{13}\} &= -\left(2\omega^2 - \frac{1}{4R^4}\right)\mathcal{L}_1 + \frac{1}{R^2}(\mathcal{L}_1\mathcal{D}_{11} + \mathcal{L}_2\mathcal{D}_{12} + \mathcal{L}_3\mathcal{D}_{13}), \\ \{\mathcal{D}_{12}, \mathcal{D}_{23}\} &= \left(2\omega^2 - \frac{1}{4R^4}\right)\mathcal{L}_2 + \frac{1}{R^2}(\mathcal{L}_1\mathcal{D}_{12} + \mathcal{L}_2\mathcal{D}_{22} - \mathcal{L}_3\mathcal{D}_{23}), \\ \{\mathcal{D}_{13}, \mathcal{D}_{23}\} &= -\left(2\omega^2 - \frac{1}{4R^4}\right)\mathcal{L}_3 + \frac{1}{R^2}(-\mathcal{L}_1\mathcal{D}_{13} + \mathcal{L}_2\mathcal{D}_{23} - \mathcal{L}_3\mathcal{D}_{33}). \end{aligned}$$

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